WAVES CAUSED BY A SUDDEN CRACK IN A CONTINUOUS ELASTIC MEDIUM

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A study is made of waves caused by a sudden crack in an elastic medium under stress. An exact solution is found for the plane problem, when the crack occurs along a strip of width l. The results are applied to an asymptotic investigation of the three-dimensional problem.

Reference [1] contains an approximate solution of the problem considered below. Maue [2] considered a related problem for a semi-infinite crack.

1. A sudden break occurs along an infinitely long plane strip of width l in an elastic medium which is in a state of plane stress. The state of stress before the occurrence of the crack is assumed to be uniform. The normal stress and normal displacement on the surface of crack are continuous (the crack does not open), while the tangential stress vanishes, so that the crack undergoes a tangential displacement.

We place the origin of the coordinates on the right edge of the crack (Fig. 1), the z-axis being directed along its edge, while the y-axis is normal to its plane.

The problem of finding the displacement field engendered by such a crack reduces to solving the equations for the displacement potentials

$$\Delta\varphi - \frac{1}{a^2}\frac{\partial^2\varphi}{\partial t^2} = 0, \qquad \Delta\psi - \frac{1}{b^2}\frac{\partial^2\psi}{\partial t^2} = 0 \qquad \left(a^2 = \frac{\lambda + 2\mu}{\rho}, \quad b^2 = \frac{\mu}{\rho}\right) \qquad (1.1)$$

where λ and μ are the Lamé coefficients and ρ is the density. The initial conditions are zero; the boundary conditions for $y = 0$ and

$-l < x < 0$ have the form

$$\sigma_{yy}^{+} = \sigma_{yy}^{-}, \qquad v^{+} = v^{-}, \qquad \tau_{xy}^{+} = \tau_{xy}^{-} = \tau = \text{const} \tag{1.2}$$

The indices plus and minus denote the corresponding limiting values of the stress and displacement components in the upper and lower half-planes for $y = 0$. The shear stress acting on the strip $-l < x < 0$ before the occurrence of the crack is denoted by τ. For the displacements u and v along the x- and y-axes we have

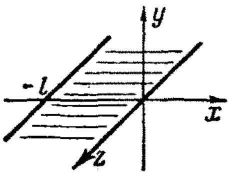

Fig. 1.

$$u = \frac{\partial\varphi}{\partial x} - \frac{\partial\psi}{\partial y}, \qquad v = \frac{\partial\varphi}{\partial y} + \frac{\partial\psi}{\partial x}$$

The boundary value problem (1.2) for equations (1.1) in the exterior of the cut is easily transformed into a problem for the upper half-plane $y \geqslant 0$ with the following conditions at $y = 0$

$$\sigma_{yy} = 0 \quad (-\infty < x < \infty), \quad \tau_{xy} = 0 \ (-l < x < 0), \quad u = u_0 \equiv \frac{b\tau}{\mu} t \quad (x < -l, \ x > 0) \tag{1.3}$$

For this the obvious symmetry properties of the displacement field in the present problem

$$u^{+}(x, y) = -u^{-}(x, -y), \qquad v^{+}(x, y) = v^{-}(x, -y) \tag{1.4}$$

should be used from the beginning and then the solution represented as the sum of a plane transverse wave propagating upwards and an auxiliary solution which also must satisfy (1.3).

The problem (1.1) and (1.3) is a particular case of the problem of diffraction of a plane wave by a slot of finite width. The solution for a semi-infinite slot was obtained by Filippov, Friedman, and Maue [3-5].

2. At the instant of cracking the front of a plane transverse wave leaves the strip, while on the edges longitudinal and transverse waves with cylindrical fronts are formed (Fig. 2). In the construction of a wave formed at one of the edges, it is clearly unnecessary to consider the second edge for $t < l/a$. Hence, for that interval of time it suffices to obtain a solution of equation (1.1) for zero initial conditions and boundary conditions of the form

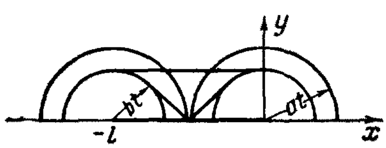

Fig. 2.

$$\sigma_{yy} = 0 \quad (-\infty < x < \infty), \quad \tau_{xy} = 0 \quad (x < 0), \quad u = u_0 \quad (x > 0) \tag{2.1}$$

We denote this solution by φ_1 and ψ_1 and the corresponding displacements by u_1 and v_1.

For $t > l/a$ the cylindrical waves formed at the edges proceed along the surface of the crack and slide off it, generating new waves. The fronts of the secondary waves near the right edge of the crack, which are excited by the waves from the left edge, are depicted in Fig. 3.

The displacements in the neighborhood of the right edge are represented as the sum of three fields: the displacement field of the primary wave from the right edge, the field of the primary wave from the left edge, and the displacement field of the secondary wave. The construction of the last may be carried out without consideration of the left edge. For this it is necessary to obtain a solution of (1.1) satisfying the conditions (2.1), where in place of u_0 one must write $-u_1(-x-l, t)$. Exactly the same thing occurs near the left edge of the cut. For $t < 2l/a$ the displacements in the medium will be described by a plane wave, and by the potentials of the primary and secondary waves from both edges. For $t > 2l/a$ waves of the third order arise, and so forth. The construction of waves of higher orders is similar to the construction of secondary waves. Thus the problem formulated above reduces to successively finding solutions of system (1.1) satisfying conditions of type (2.1), where in place of u_0 there is a function of the coordinates and time corresponding to horizontal displacements in the wave arriving from the opposite edge.

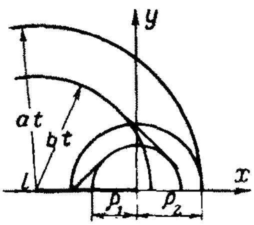

Fig. 3.

These solutions are constructed in a similar manner as in [6,7]. Applying the two-sided Laplace transform with respect to x and the one-sided Laplace transform with respect to t, for which the transform variables are q and p, respectively, one obtains from (1.1) and (2.1) functional equations relating the transforms of the tangential stress τ_{xy} and displacement u for $y = 0$. The Wiener-Hopf-Fok method [8] is used to find from these equations the transform of the displacement u and to construct the transforms of the potentials for $y \geqslant 0$. We omit the calculations, which are similar to those in [6,7]. In view of the fact that completely identical waves are generated at the opposite edges, we restrict ourselves to the consideration of waves arising at the right edge of the cut. The transforms of the potentials of the first longitudinal and transverse waves have the form

$$\Phi_1(y, q, p) = 2m_0 \frac{\tau b}{\rho p^4} M(s) \exp\left(-\frac{py}{b}\sqrt{\gamma^2 - s^2}\right) \qquad \left(\gamma = \frac{b}{a}, \quad s = \frac{bq}{p}\right)$$

$$\Psi_1(y, q, p) = -m_0 \frac{\tau b}{\rho p^4} \frac{2s^2-1}{s\sqrt{1-s^2}} M(s) \exp\left(-\frac{py}{b}\sqrt{1-s^2}\right)$$

$$\left(m_0 = \frac{1}{\sqrt{2(1-\gamma^2)}}\right) \qquad (2.2)$$

Here we denote

$$M(s) = \frac{\sqrt{1-s}}{\vartheta - s} e^{g(s)}, \qquad g(s) = \frac{1}{\pi}\int_\gamma^1 \frac{\varphi(\xi)\, d\xi}{\xi - s}$$

$$\varphi(\xi) = \tan^{-1} \frac{4\xi^2 \cdot \sqrt{(1-\xi^2)(\xi^2-\gamma^2)}}{(2\xi^2-1)^2} \qquad (2.3)$$

where ϑ is the root of the Rayleigh equation

$$G(s) \equiv (2s^2-1)^2 + 4s^2\sqrt{(1-s^2)(\gamma^2-s^2)} = 0$$

The following expressions are obtained for the transforms of the potentials of the waves generated for $t > (n-1)l/a$:

$$\Phi_n = s\Phi_1 W_{n-1}(s, pl/b), \qquad \Psi_n = s\Psi_1 W_{n-1}(s, pl/b) \qquad (2.4)$$

where Φ_1 and Ψ_1 are defined by formulas (2.2), while W_m is defined by the relation

$$W_m = \left(\frac{-1}{\pi}\right)^m \int_{v_m} \Pi_m(M_m) \exp\left(-\frac{pl}{b}\sum_{k=1}^m s_k\right) \frac{dv_m}{s_m + s} \qquad (2.5)$$

The integration is performed over the volume $s_k \geqslant \gamma$, where $k = 1, \ldots, m$. Furthermore

$$\Pi_m(M_m) = \prod_{k=1}^m \frac{P(s_k)}{s_k + s_{k-1}} \qquad (s_0 = 0)$$

$$P(s) = \sqrt{s-\gamma}\, P_1(s) \qquad (\gamma \leqslant s \leqslant 1) \qquad (2.6)$$

$$P(s) = \sqrt{s-\gamma}\, P_1(s) + \sqrt{s-1}\, P_2(s) \qquad (s > 1)$$

where

$$P_1(s) = 8(1-\gamma^2) \frac{s^2(1-s)(\vartheta+s)^2 \sqrt{s+\gamma}}{(2s^2-1)^4 + 16s^4(1-s^2)(s^2-\gamma^2)} e^{-2g(-s)}$$

$$P_2(s) = -2(1-\gamma^2) \frac{(2s^2-1)^2(\vartheta+s)^2}{[(2s^2-1)^4 + 16s^4(1-s^2)(s^2-\gamma^2)]\sqrt{s+1}} e^{-2g(-s)}$$

3. From (2.4) and (2.5) it follows that in order to obtain the inverse transforms of Φ_n and Ψ_n, it is sufficient to perform the inversion with respect to q and p for the expressions

$$\frac{s\Phi_1}{s_{n-1}+s}\exp\left(-\frac{pl}{b}\sum_{k=1}^{n-1}s_k\right), \qquad \frac{s\Psi_1}{s_{n-1}+s}\exp\left(-\frac{pl}{b}\sum_{k=1}^{n-1}s_k\right) \tag{3.1}$$

Such a procedure corresponds to performing the inversion under the integral sign in formula (2.5), where the integrand depends on s and p. However, it is convenient to obtain the inverse transforms of the expressions obtained from (3.1) by multiplying by p^3. This means that the third derivatives of the unknown functions with respect to time will be found.

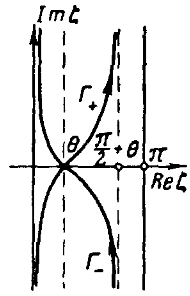

Fig. 4.

We denote the inversion integrals with respect to q of the expressions mentioned above by A_n° and B_n°, respectively, and the complete inverse transforms of these expressions by A_n and B_n.

Transforming to a new variable of integration ζ in the integral A_n°, such that $q = -a^{-1}p\cos\zeta$, and introducing polar coordinates r, θ in place of the Cartesian coordinates x and y, we obtain

$$A_n^{\circ} = \exp\left(-\frac{pl}{b}\sum_{k=1}^{n-1}s_k\right)\frac{1}{2\pi i}\int_{\Gamma}C_n(\zeta)\exp\left[-\frac{pr}{a}\cos(\zeta-\theta)\right]d\zeta \tag{3.2}$$

$$C_n(\zeta) = -m_0\gamma\,\frac{\tau}{p}\,\frac{\sin 2\zeta}{s_{n-1}-\gamma\cos\zeta}\,M(-\gamma\cos\zeta)$$

The contour Γ is located in the strip $0 < \operatorname{Re}\zeta < \pi$ such that the difference between the abscissa of a point on it and the point $\zeta = \theta$ is less than $\pi/2$. Moreover, its position depends on the sign of $\operatorname{Im} p$: in order that the inequality

$$\operatorname{Re}\,[p\cos(\zeta-\theta)] > 0 \tag{3.3}$$

hold on the entire contour, for $\operatorname{Im} p > 0$ the contour Γ must coincide with Γ_+, while for $\operatorname{Im} p < 0$ it must coincide with Γ_-, as depicted in Fig. 4. Carrying out the inversion with respect to p, we obtain

$$A_n = \frac{1}{2\pi i}\int_{-i\infty}^{0}dp\,\frac{1}{2\pi i}\int_{\Gamma_-}C_n(\zeta)\,e^{p\Delta}\,d\zeta + \frac{1}{2\pi i}\int_{0}^{i\infty}dp\,\frac{1}{2\pi i}\int_{\Gamma_+}C_n(\zeta)\,e^{p\Delta}\,d\zeta$$

$$\Delta = t - \frac{r}{a}\cos(\zeta-\theta) - \frac{l}{b}\sum_{k=1}^{n-1}s_k \tag{3.4}$$

The fulfillment of condition (3.3) makes it possible to interchange the order of integration with respect to p and ζ. The expression obtained after this operation may be transformed into an integral over a closed contour Γ, having the form of a figure-eight (Fig. 5)

$$A_n = -\frac{1}{4\pi^2}\int_\Gamma C_n(\zeta)\frac{d\zeta}{\Delta}$$

The integrand vanishes at infinity and has in the strip only poles, which coincide with the zeros of the function Δ given by formula (3.4). For $at - r < (n - 1)l$ these zeros lie on the real axis and do not fall within the contour Γ. Hence $A_n = 0$.

For $at - r > (n - 1)l$ there are two zeros of the function Δ located on the straight line $\operatorname{Re}\zeta = \theta$ at conjugate points. The coordinates of the zero lying in the upper half-strip is given by the formula

$$\zeta_0 = \theta + i\ln\left(\frac{aT}{r} + \sqrt{\frac{a^2T^2}{r^2} - 1}\right) \qquad \left(T = t - \frac{l}{b}\sum_{k=1}^{n-1} s_k\right) \tag{3.5}$$

Hence the integral defining A_n is equal to the sum of the residues at these points

$$A_n = -\frac{a}{\pi}\frac{1}{\sqrt{a^2T^2 - r^2}}\operatorname{Re} C_n(\zeta) \tag{3.6}$$

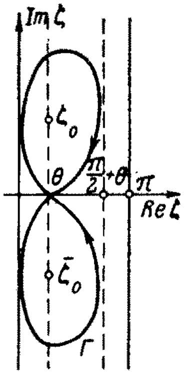

Fig. 5.

Hence we obtain from (2.4) the inverse transform of the function Φ_n, which we will denote by φ_n

$$\varphi_n(r, \theta, t) = 0 \qquad (at - r < (n-1)l)$$

$$\frac{\partial^3\varphi_n}{\partial t^3} = \left(\frac{-1}{\pi}\right)^n a \int_{v_{n-1}} \Pi_{n-1}(M_{n-1})\frac{\operatorname{Re} C_n(\zeta_0)}{\sqrt{a^2T^2 - r^2}}\,dv \qquad (at - r > (n-1)l) \tag{3.7}$$

The integration is carried out over the volume

$$\sum_{k=1}^{n-1} s_k \leqslant (at - r)/\gamma l, \qquad s_k \geqslant \gamma \quad (k = 1, \ldots, n-1) \tag{3.8}$$

The inverse transform of the function Ψ_n is found in a similar manner, but in this one runs into some complications arising from the presence in the strip of singularities of fractional order and branch cuts, which are connected with the leading waves. For the potential of the cylindrical waves we obtain

$$\frac{\partial^3\psi_n}{\partial t^3} = \left(\frac{-1}{\pi}\right)^n b \int_{v_{n-1}} \Pi_{n-1}(M_{n-1})\operatorname{Re} D_n(\eta_0)\frac{dv}{\sqrt{b^2T^2 - r^2}} \qquad (bt - r > (n-1)\gamma l) \tag{3.9}$$

The integration is carried out over the volume

$$\sum_{k=1}^{n-1} s_k \leqslant \frac{bt-r}{l}, \qquad s_k \geqslant \gamma \qquad (k=1,\ldots, n-1) \tag{3.10}$$

In (3.9) the following notation was introduced

$$\eta_0 = \theta + i \ln \left[\frac{bT}{r} + \left(\frac{b^2T^2}{r^2} - 1\right)^{1/2}\right], \quad D_n(\eta) = \frac{m_0\tau}{\rho} \frac{\cos 2\eta}{s_{n-1} - \cos\eta} M(-\cos\eta) \tag{3.11}$$

where T is defined by (3.5). A similar method of calculating the inversion integral appears in [9].

The foregoing analysis enables one to write the potential of the displacement field of the longitudinal waves at an arbitrary instant t and an arbitrary point r, θ (the origin of the coordinates is at the right edge) in the form

$$\varphi(r, \theta, t) = \sum_{k=1}^{N_1} \varphi_k(r, \theta, t) + \sum_{k=1}^{N_2} \varphi_k(r_1, \pi - \theta_1, t)$$

where r_1 and θ_1 are the polar coordinates of the same point in a coordinate system fixed at the left edge of the cut, and

$$N_1 = E\left(\frac{at-r}{l}\right) + 1, \qquad N_2 = E\left(\frac{at-r_1}{l}\right) + 1$$

Here $E(x)$ is the integral part of x.

The potential field of the transverse waves has a similar form, if the leading wave is not considered. Thus the problem formulated in Section 1 is solved. We will now investigate the results obtained.

4. We consider the potentials of the first longitudinal and transverse waves generated at the right edge of the crack. In accordance with (3.7) and (3.9) we have

$$\frac{\partial^3\varphi_1}{\partial t^3} = -\frac{2\gamma m_0}{\pi}\frac{a\tau}{\rho}\frac{1}{\sqrt{a^2t^2 - r^2}} \operatorname{Re}\,[\sin\zeta_0\, M(-\gamma\cos\zeta_0)]$$

$$\frac{\partial^3\psi_1}{\partial t^3} = \frac{m_0}{\pi}\frac{b\tau}{\rho}\frac{1}{\sqrt{b^2t^2 - r^2}} \operatorname{Re}\left[\frac{\cos 2\eta_0}{\cos\eta_0} M(-\cos\eta_0)\right] \tag{4.1}$$

$$\left(\zeta_0 = \theta + i\ln\left(\frac{at}{r} + \sqrt{\frac{a^2t^2}{r^2} - 1}\right), \quad \eta_0 = \theta + i\ln\left(\frac{bt}{r} + \sqrt{\frac{b^2t^2}{r^2} - 1}\right)\right)$$

These formulas may also be obtained from the solution of the problem of diffraction of a transverse wave on the half-plane [3,4]. The frontal zone of the longitudinal wave corresponds to the relation

$$1 \gg (at - r) / r$$

Thus $\zeta_0 = \theta$ is accurate to within small quantities of the order of $\sqrt{[(at - r)/r]}$, and from (4.1) it follows that

$$\varphi_1 (r, \theta, t) = \frac{8 \sqrt{2}\, \gamma m_0}{15 \pi} \frac{\tau}{a^2 p} \sin \theta M (-\gamma \cos \theta) \frac{(at - r)^{5/2}}{\sqrt{r}} \qquad (4.2)$$

The dependence of the radial displacement $u_r = \partial\varphi_1/\partial r$ on θ in the frontal zone is shown in Fig. 6, where the quantity

$$U = \frac{4 \sqrt{2}\, \gamma m_0}{3\pi} \sin \theta M (-\gamma \cos \theta)$$

proportional to u_r, is laid off along the radii of the circle corresponding to the front of the longitudinal wave.

In a similar manner the expansion near the front of the transverse wave yields

$$\psi_1 (r, \theta, t) = \frac{4 \sqrt{2} m_0}{15 \pi} \frac{\tau}{b^2 \rho} \frac{\cos 2\theta}{\cos \theta} K (\theta) \frac{(bt - r)^{5/2}}{\sqrt{r}} \qquad (4.3)$$

where

$$K (\theta) = M (-\cos \theta) \qquad (0 \leqslant \theta < \pi - \cos^{-1} \gamma) \qquad (4.4)$$

$$K (\theta) = \frac{2 (1 - \gamma^2) \cos^2 2\theta \sin \theta}{[\cos^4 2\theta + 16 \sin^2 2\theta \cos^2 \theta (\cos^2 \theta - \gamma^2)]\, M (\cos \theta)} \qquad (\pi - \cos^{-1} \gamma < \theta \leqslant \pi)$$

The formula (4.3) is useless in the neighborhood of the point $\theta = \pi/2$. The required expression may be obtained from (4.1) if this point is considered separately.

The dependence of the tangential displacement $u_\theta = \partial\psi_1/\partial r$ on θ in the frontal zone of the transverse wave is shown in Fig. 6, where the quantity

$$V = - \frac{\sqrt{2}\, m_0}{3\pi\gamma^2} \frac{\cos 2\theta}{\cos \theta} K (\theta)$$

proportional to u_θ, is laid off along the radii of the circle corresponding to the front of the transverse wave.

If the distance to the field point greatly exceeds the width l of the crack, the cylindrical waves generated on the edges may approximately be considered to originate from a single center. The dependence of the displacements on the angle θ (in the previous coordinate system) in such a combined cylindrical wave may be constructed with the aid of formulas (4.2) and (4.3) if it is assumed that for $|\theta| < \pi/2$ the displacements coincide with the displacements in the wave coming from the right edge of the crack, while for $\pi < 2\theta < 3\pi$ they coincide with the displacements in the wave from the left edge. Such a dependence for the radial component u_r of the first longitudinal wave and the tangential component u_θ of the transverse wave is shown in Fig. 7. It is essential to note that the radial displacements in the first longitudinal wave are zero for $\theta = 0$, $\theta = \pi/2$ and $\theta = \pi$, while the tangential displacements in the transverse wave are zero for $\theta = \pm\,\pi/4$ and $\theta = \pm\,3\pi/4$.

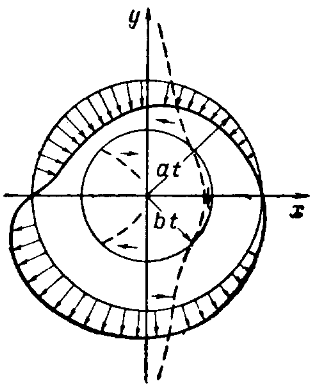

Fig. 6.

If it is assumed that the crack considered here is a model of a seismic center, then the above properties of the displacement field coincide with the properties of the models of Khodno-Vvedenskaia [10] and others and do not agree with the corresponding properties of the Keilis-Borok model [11].

5. The expansion near the front of an arbitrary wave propagating from the right edge of the crack may be obtained from formulas (3.7) and (3.9), just as in Section 4. We consider φ_n as given by (3.7). We denote by h_0 the distance measured inward from the first front of this wave

$$h_0 = at - r - (n-1)\,l \qquad (5.1)$$

Also, we let $1 \gg h_0/l$, corresponding to the region near the wavefront. Then from (3.8) it follows that

$$\sum_{k=1}^{n-1} s_k < \gamma\,(n + h_0\,/\,l)$$

Using this, with the aid of (3.5) it may be shown that $\zeta_0 = \theta$ to an accuracy within small quantities of order $\sqrt{(h_0/l)}$. Moreover, with the same accuracy the result

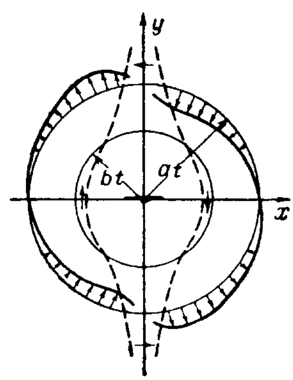

Fig. 7.

$$\Pi_{n-1} = \gamma\left[\frac{P_1(\gamma)}{2\gamma}\right]^{n-1}\left[\prod_{k=1}^{n-1}(s_k-\gamma)\right]^{1/2}$$

may be obtained from (2.6).

Retaining only the dominant terms in (3.7), we obtain the following formula:

$$\varphi_n(r, \theta, t) = f_0(n) \frac{l^2\tau}{a^2\rho} \frac{\sin 2\theta}{1-\cos\theta} M(-\gamma\cos\theta) \frac{\sqrt{l}}{\sqrt{r}} \left(\frac{h_0}{l}\right)^{1+3n/2} \qquad (5.2)$$

$$f_0(n) = -\left(\frac{-1}{2\pi}\right)^n m_0 2\sqrt{2}\,\gamma^{\frac{n+1}{2}} \frac{[P_1(\gamma)\,\Gamma(^3/_2)]^{n-1}}{\Gamma(^1/_2)\,\Gamma(1+3n/2)}$$

However, formula (3.7) describes not only the wave with the front $h_0 = 0$, the neighborhood of which was just investigated, but also the waves with fronts

$$h \equiv at - r - (n-m-1)\,l - ml/\gamma = 0 \qquad (m=0,1,\ldots,n-1) \qquad (5.3)$$

These fronts are caused by the waves which propagate from the edges of the crack after having passed over the crack $n - m - 1$ times as longitudinal waves and m times as transverse waves. For fixed n and m there will be C_{n-1}^m of such waves, but all of them arise simultaneously and will have a common front. From among them C_{n-2}^m will be chosen, which passed on the last stage as longitudinal waves and posses identical expansions near the wavefront which do not coincide with the expansions of the waves which passed as transverse waves on the last stage. The latter waves also have identical expansions.

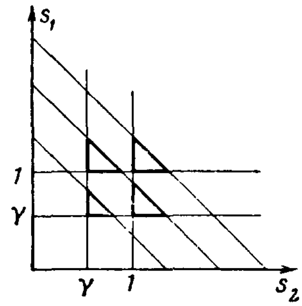

Fig. 8.

We consider the case $n = 3$ as an illustration. From (2.6) we have

$$s_1(s_1+s_2)\,\Pi_2 = P_1(s_1)\,P_1(s_2)\sqrt{(s_1-\gamma)(s_2-\gamma)} + P_1(s_1)\,P_2(s_2)\sqrt{(s_1-\gamma)(s_2-1)} + P_1(s_2)\,P_2(s_1)\sqrt{(s_1-1)(s_2-\gamma)} + P_2(s_1)\,P_2(s_2)\sqrt{(s_1-1)(s_2-1)}$$

Hence the integral in (3.7) breaks up into the sum of four integrals. From (2.6) it follows that the first differs from zero for $at - r > 2l$, the second and third for $at - r > l(1 + 1/\gamma)$, and the fourth for $at - r > 2l/\gamma$. These integrals describe only the potentials of the waves in question. The construction of the frontal expansions is carried out by means of integrations over small triangular areas with vertices at the points (γ, γ), $(1, \gamma)$, $(\gamma, 1)$, $(1, 1)$ as shown in Fig. 8.

The case of arbitrary n and m is treated similarly. We have

$$\varphi_n(r, \theta, t) = \sum_{m=0}^{n-1} \varphi_{nm}(r, \theta, t)$$

where

$$\varphi_{nm} = f_m(n) \frac{l^2\tau}{a^2\rho} \sin 2\theta \, M(-\gamma \cos\theta) \times \times \left(\frac{c_{nm}}{1-\gamma\cos\theta} + \frac{d_{nm}}{1-\cos\theta}\right) \sqrt{\frac{l}{r}} \left(\frac{h_m}{l}\right)^{1+3n/2}$$

$$f_m(n) = -\left(\frac{-1}{\pi}\right)^n \frac{m_0}{\sqrt{2\gamma}} \gamma^{3n/2} \frac{P_1^{n-m-1}(\gamma) P_2^m(1) \Gamma^{n-1}(3/2)}{\Gamma(1/2)\Gamma(1+3n/2)} \tag{5.4}$$

The values of h_m are given by formula (5.3); in addition, c_{nm} denotes the sum of all possible products

$$\frac{1}{1+s_{n-2}} \prod_{k=1}^{n-2} \frac{1}{s_k + s_{k-1}} \qquad (s_0 = 0)$$

where m of the quantities $s_1, \ldots, s_{n-2}$ are equal to 1, while the rest equal γ.

Similarly, d_{nm} is the sum of all possible products

$$\frac{1}{\gamma(\gamma + s_{n-2})} \prod_{k=1}^{n-2} \frac{1}{s_k + s_{k-1}}$$

where $m - 1$ of the quantities $s_1, \ldots, s_{n-2}$ equal 1, and the rest equal γ.

From the above formulas it follows that the order of the discontinuity at the front of all the waves corresponding to the potential φ_n is less than the discontinuity of the waves with the potential φ_{n-1} by 3/2. Moreover, the coefficients $f_m(n)$ rapidly decrease with increasing n. Thus the waves subjected to repeated diffraction are strongly damped. It is interesting to note that the frontal expansions of all longitudinal waves, starting with the second, have the factor sin 2θ. Hence it follows that when repeatedly diffracted waves are registered at great distances, the radial components of the displacement in the frontal zones will be equal to zero on the line which is an extension of the crack and on the perpendicular line passing through the middle of the crack.

The frontal zones of multiply diffracted transverse waves, which are described by equations (3.9) to (3.11), are investigated in a similar manner. As before, the leading wave is excluded from consideration

$$\psi_n(r, \theta, t) = \sum_{m=0}^{n-1} \psi_{nm}(r, \theta, t) \tag{5.5}$$

$$\psi_{nm} = -f_m(n) \frac{l^2\tau}{a^2\rho} \cos 2\theta\, K(\theta) \left(\frac{c_{nm}}{1-\cos\theta} + \frac{\gamma d_{nm}}{\gamma - \cos\theta} \right) \sqrt{\frac{l}{r}} \left(\frac{h_m}{l} \right)^{1+3n\ \ 2}$$

where

$$h_m = bt - r - \gamma l\,(n - m - 1) - ml$$

The coefficients $f_m(n)$, c_{nm} and d_{nm} have the same values as in (5.4). The function $K(\theta)$ is given by formula (4.4).

The formula for ψ_{nm} becomes meaningless at the points where the front $h_m = 0$ is tangent to: the front of the previous transverse wave ($\theta = 0$), the front of the leading wave corresponding to the previous transverse wave ($\theta = \cos^{-1}\gamma$), and the front of the leading wave ($\theta = \pi - \cos^{-1}\gamma$).

These singular points are shown in Fig. 3.

It is interesting to note that in all the multiply diffracted transverse waves the tangential component vanishes in the frontal region for $\theta = \pm\pi/4$ and $\theta = \pm 3\pi/4$, i.e. for the same angles as in the first wave (4.3).

6. Consideration is now given to the corresponding three-dimensional problem. The state of stress before cracking and the properties of the crack (continuity of the normal components of stress and displacement) are assumed to be the same as in Section 1, but the crack is assumed to be a disk of radius l. The exact solution of such a problem is not known. However, using the results presented above, we may obtain an asymptotic representation of the displacement field near the fronts of the first longitudinal and transverse waves.

We denote the constant tangential stress acting on the disk before cracking by τ_0. We introduce a system of coordinates α, r, θ. The surface α = const is a half-plane perpendicular to the disk and passing through its center. The angle between the line of intersection of the half-plane and the disk and the stress τ_0 is denoted α. We place the origin of the coordinate system r, θ on the half-plane α = const, at the point of its intersection with the edge of the crack. The angle θ will be measured from the extension of the radius drawn from the center of the disk to the origin of the system of polar coordinates (Fig. 9).

At the instant of cracking a plane transverse wave leaves the disk, while at the edge there arise longitudinal and transverse waves with toroidal fronts

$$r = at, \qquad r = bt \tag{6.1}$$

(the time t is measured from the instant of cracking), which will be joined by the conical front of the primary wave. A section of the set of fronts made by a plane perpendicular to the disk and passing through its center is shown in Fig. 2.

Let $at \ll l$. We consider the displacements in a narrow slice cut out of the region of disturbance by the two sides of the angle $d\alpha$. For their approximate calculation it is necessary to know the solution of the problem when the half-plane $y = 0$, $x < 0$ (Fig. 10) is freed from the tangential stresses $\tau_{xy} = \tau_0 \cos\alpha$ and $\tau_{zy} = -\tau_0 \sin\alpha$ as a result of the crack. This problem is split into two problems: the first concerning waves in which the displacements are parallel to the plane $z = 0$, and its solution given by formulas (4.1) to (4.3); the second concerning waves with displacements parallel to the z-axis. The latter reduces to a well-known problem, and the displacements in the frontal region of the cylindrical wave have the form

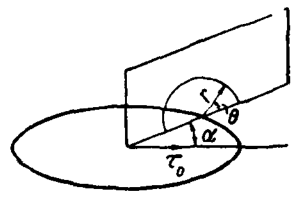

Fig. 9.

$$w = \frac{4}{3\pi} \frac{\tau_0 \sin\alpha}{\rho b^2} \frac{\sin\theta/2}{\cos\theta} \frac{(bt - r)^{3/2}}{\sqrt{r}} \tag{6.2}$$

Thus formulas (6.2), (4.2) and (4.3)(setting $\tau = \tau_0 \cos\alpha$ in the latter) give the frontal expansions of all toroidal waves for $at \ll l$. But if the frontal expansion of any wave is known at a certain fixed instant, then this expansion may be constructed for an arbitrary position of the front provided that the rays have neither singular lines nor singular points (caustics and focuses) [12]. The fronts of the toroidal waves satisfy these requirements for $|\theta| < \pi/2$. Accordingly [12], the principal parts of the expansions of u_0 and u at the points of intersection of a single ray with two different positions of the front are related by the formula

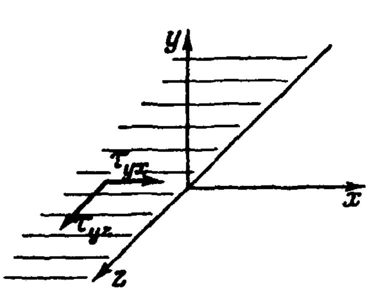

Fig. 10.

$$u = u_0 \sqrt{R_1^\circ R_2^\circ / R_1 R_2} \tag{6.3}$$

where R_1°, R_2° and R_1, R_2 are the principal radii of curvature of the fronts at the points of intersection with the chosen ray. It can be shown that for the toroidal fronts (6.1)

$$\frac{R_1^\circ R_2^\circ}{R_1 R_2} = \frac{r_0 (r_0 \cos\theta + l)}{r (r \cos\theta + l)} \tag{6.4}$$

where r_0 is the radius of the initial position of the toroidal front.

Using (4.2) and (6.4) for the radial components of the displacement in the frontal region of the longitudinal wave, we find for $|\theta| < \pi/2$

$$u_0 = \frac{4\sqrt{2}\,\gamma m_0}{3\pi} \frac{\sqrt{l}\,\tau_0}{a^2\rho} \cos\alpha \sin\theta\, M(-\gamma\cos\theta) \frac{(at - r)^{3/2}}{\sqrt{r(r\cos\theta + l)}} \tag{6.5}$$

For the displacement components along the meridian and latitude in the frontal regions of the transverse wave we obtain from (4.3), (6.2) and (6.4)

$$u_\theta = -\frac{2\sqrt{2}\,m_0}{3\pi} \frac{\sqrt{l}\,\tau_0}{b^2\rho} \frac{\cos\alpha\cos 2\theta}{\cos\theta} K(\theta) \frac{(bt - r)^{3/2}}{\sqrt{r(r\cos\theta + l)}} \tag{6.6}$$

$$u_\alpha = \frac{4}{3\pi} \frac{\sqrt{l}\,\tau_0}{b^2\rho} \frac{\sin\alpha\sin\theta/2}{\cos\theta} \frac{(bt - r)^{3/2}}{\sqrt{r(r\cos\theta + l)}} \tag{6.7}$$

If $bt \gg l$ then the external portion ($|\theta| < \pi/2$) of any of the toroidal fronts may be considered to be approximately a sphere with center at the middle of the disk on which the crack occurs. The coordinates r, θ, α introduced above may, with the same accuracy, be treated as spherical coordinates with origin at the center of the disk (the angle θ is measured from the plane of the crack). Then, neglecting l in comparison with r in (6.5) and (6.7), we obtain

$$\begin{aligned} u_r &= \frac{4\sqrt{2}\gamma m_0}{3\pi} \frac{\sqrt{l}\,\tau_0}{a^2\rho} \frac{\cos\alpha\sin\theta}{\sqrt{\cos\theta}} M(-\gamma\cos\theta) \frac{(at - r)^{3/2}}{r} \\ u_\theta &= -\frac{2\sqrt{2}\,m_0}{3\pi} \frac{\sqrt{l}\,\tau_0}{b^2\rho} \frac{\cos\alpha\cos 2\theta}{(\cos\theta)^{3/2}} K(\theta) \frac{(bt - r)^{3/2}}{r} \\ u_\alpha &= \frac{4}{3\pi} \frac{\sqrt{l}\,\tau_0}{b^2\rho} \frac{\sin\alpha\sin\theta/2}{(\cos\theta)^{3/2}} \frac{(bt - r)^{3/2}}{r} \end{aligned} \tag{6.8}$$

These approximate formulas are not applicable for $\theta = \pm\pi/2$. It is interesting to note that in the frontal zone of the longitudinal wave $u_r = 0$ on the meridians $\alpha = \pi/2$ and $\alpha = 3\pi/2$, as well as on the equator $\theta = 0$. In the transverse wave $u_\theta = 0$ on the same meridians and on the two circles of latitudes $\theta = \pm\pi/4$. The displacement u_Ω is zero on the meridians $\alpha = 0$ and $\alpha = \pi$.

If it is assumed that the crack considered here is a model of a seismic center, then the properties of the displacement field cited above agree with the properties of the Khodno-Vvedenskaia model [10] but disagree with the properties of the Keilis-Borok model [11].

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